

AN EFFECTIVE CARATHÉODORY THEOREM

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ABSTRACT. By means of the property of effective local connectivity, the computability of finding the Carathéodory extension of a conformal map of a Jordan domain onto the unit disk is demonstrated.

1. INTRODUCTION

In 1851, B. Riemann proved his powerful Riemann Mapping Theorem which states that every simply connected subset of the plane is conformally equivalent to the unit disk, \mathbb{D} . E. Bishop gave a constructive proof of this theorem in 1967 (now in [1]), and in 1999 P. Hertling proved a uniformly effective version of this theorem [9].

In classical complex analysis, the next step up from the Riemann Mapping Theorem is the Carathéodory Theorem which states that a conformal map of a simply connected Jordan domain onto the unit disk can be extended to a homeomorphism of the closure of the domain with the closure of the disk. Such an extension is called the *boundary extension* or *Carathéodory extension*. This result is the basis for demonstrating the existence of solutions to Dirichlet problems on simply connected Jordan domains.

Here, we will extend Hertling's work by proving a uniformly effective version of the Carathéodory Theorem. That is, roughly speaking, we show that from sufficiently good approximations to a parameterization of a Jordan curve γ , we can compute arbitrarily good approximations of a conformal map of the interior of γ onto the unit disk as well as arbitrarily good approximations of its boundary extension. This will be achieved by means of *effective local connectivity*. This concept, which first appeared in [11], is the subject of a previous investigation by Daniel and McNicholl [5] and also plays an important role in [2]. The resulting construction of the Carathéodory extension can be considered as a new proof of this result.

The paper is organized as follows. In Sections 2 and 3, we summarize background information from complex analysis and computable analysis. In Section 4, we head straight for the proof of the main theorem.

2. BACKGROUND FROM COMPLEX ANALYSIS

Most of our terminology and notation from complex analysis can be found in [3] and [4].

We will use $C[0, 1]$ to denote the set of continuous functions from $[0, 1]$ into \mathbb{C} .

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When $X \subseteq \mathbb{C}$, define

$$D_\epsilon(X) = \bigcup_{p \in X} D_\epsilon(p).$$

Let $\|\cdot\|_\infty$ denote the L^∞ norm on $C[0, 1]$. Hence, if $\|f - g\|_\infty < \epsilon$, then $\text{ran}(f) \subseteq D_\epsilon(\text{ran}(g))$.

An $f \in C[0, 1]$ is *polygonal* if there is a partition of $[0, 1]$, $0 = a_0 < a_1 < \dots < a_{k-1} < a_k = 1$, such that f is linear on each of $[a_j, a_{j+1}]$. f is *rational polygonal* if each a_j is a rational number and each $f(a_j)$ is a rational point. Such a function is completely determined by the tuple $(a_0, f(a_0), \dots, a_k, f(a_k))$.

We say that an arc A_2 *extends* an arc A_1 if $A_2 \supset A_1$ and A_2, A_1 have an endpoint in common.

A *parameterization* of an arc A is a homeomorphism of $[0, 1]$ with A . A parameterization of a Jordan curve J is a homeomorphism of $\partial\mathbb{D}$ with J . We will follow the usual convention of identifying an arc or Jordan curve with any of its parameterizations.

A *domain* is an open, connected subset of the plane. A domain is *Jordan* if it is bounded by Jordan curves.

A function u on a domain D is *harmonic* if

$$\frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0$$

for all $z \in D$.

Suppose D is a Jordan domain and that f is a bounded piecewise continuous function on its boundary. The resulting *Dirichlet problem* is to find a harmonic function u on D with the property that

$$\lim_{z \rightarrow \zeta} u(z) = f(\zeta)$$

for all $\zeta \in \partial D$ at which f is continuous. Solutions to Dirichlet problems always exist and are unique. The function f is said to provide the *boundary data* for this problem.

3. BACKGROUND FROM COMPUTABLE ANALYSIS AND COMPUTABILITY

An informal summary of the fundamentals of Type-Two Effectivity appears in [5]. We identify \mathbb{C} with \mathbb{R}^2 . Hence, we can also use the naming systems in [5] for \mathbb{C} and its hyperspaces. We shall identify objects with their names wherever this results in simplicity of exposition while not creating misconceptions.

While we identify arcs and Jordan curves with their parameterizations, such curves will always be named by names of their parameterizations.

We also refer the reader to [5] for the definitions of *local connectivity witness* and *uniform local arcwise connectivity witness* and related theorems. In addition to these theorems, we will need the following.

Lemma 3.1. *From a name of a homeomorphism f of $\partial\mathbb{D}$ with a Jordan curve J , we can compute a local connectivity witness for J .*

Proof. It follows by essentially the same argument as in the proof of Theorem 6.2.7 of [13] that we can compute, uniformly in the given data, a modulus of continuity for f , m . Compute f^{-1} and a modulus of continuity for f^{-1} , m_1 . Let $h = m_1 \circ m$.

We claim that h is a local connectivity witness for J . For, let $k \in \mathbb{N}$, and let $w_1 \in J$. Let z_1 be the unique preimage of w_1 under f . Let I be the interval $(z_1 - 2^{-m(k)}, z_1 + 2^{-m(k)})$. Let $C = f[I]$. We claim that

$$J \cap D_{2^{-h(k)}}(w_1) \subseteq C \subseteq D_{2^{-k}}(w_1).$$

For, let $w_2 \in C$. Let z_2 be the unique preimage of w_2 under f . Then, $z_2 \in I$, and so $|z_1 - z_2| < 2^{-m(k)}$. Hence, $|w_2 - w_1| < 2^{-k}$. Thus, $C \subseteq D_{2^{-k}}(w_1)$.

Now, suppose $w_2 \in J \cap D_{2^{-h(k)}}(w_1)$. Again, let z_2 be the unique preimage of w_2 under f . Then, $|z_1 - z_2| < 2^{-m(k)}$. Hence, $z_2 \in I$. Thus, $w_2 \in C$. \square

Lemma 3.2. *Given names of arcs $\gamma_1, \dots, \gamma_n$ such that $\partial\mathbb{D} = \gamma_1 + \dots + \gamma_n$, and given names of continuous real-valued functions f_1, \dots, f_n such that $\gamma_j = \text{dom}(f_j)$, we can compute a name of the harmonic function u on \mathbb{D} defined by the boundary data*

$$f(\zeta) = \begin{cases} f_j(\zeta) & \zeta \in \gamma_j, \zeta \neq \gamma_j(0), \gamma_j(1) \\ \max_j \max f_j & \text{otherwise.} \end{cases}$$

In addition we can compute the extension of u to $\overline{\mathbb{D}}$ except at the endpoints of the arcs $\gamma_1, \dots, \gamma_n$.

Proof. Let u be the solution to the resulting Dirichlet problem on \mathbb{D} . For $z \in \mathbb{D}$, we use the Poisson Integral Formula

$$(3.1) \quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta \quad z \in \mathbb{D}.$$

(See, for example, Theorem I.1.3 of [6].) In the case under consideration, we have

$$u(z) = \sum_j \frac{1}{2\pi} \int_{\gamma_j} f_j(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|.$$

Since integration is a computable operator, this shows we can compute u on \mathbb{D} .

Since we are given f_1, \dots, f_n , it might seem immediate that we can now compute the extension of u to $\overline{\mathbb{D}}$ except at the endpoints of $\gamma_1, \dots, \gamma_n$. However, it is not possible to determine from a name of a point $z \in \overline{\mathbb{D}}$ if $z \in \partial\mathbb{D}$. To see what the difficulty is, and to lead the way towards its solution, we delve a little more deeply into the formalism. Suppose we are given a name of a $z \in \overline{\mathbb{D}}$, p . As we read p , it may be that at some point we find a subbasic neighborhood R whose closure is contained in \mathbb{D} . In this case, we can just use equation (3.1). However, if we keep finding subbasic neighborhoods that intersect $\partial\mathbb{D}$, then at some point we must commit to an estimate of $u(z)$. If we guess $z \in \partial\mathbb{D}$, then later this guess and this resulting estimate may turn out to be incorrect. We face a similar problem if we guess $z \in \mathbb{D}$. The heart of the matter then is to estimate the value of $u(\zeta)$ when ζ is near z and in \mathbb{D} . This can be done by effectivizing one of the usual proofs that $\lim_{\zeta \rightarrow z} u(\zeta) = f(z)$ when z is between the endpoints of a γ_j . To begin, fix rational numbers $\alpha \in [-\pi, \pi]$, $2\pi > \delta > 0$, and $0 < \rho < 1$. Let

$$S(\rho, \delta, \alpha) =_{df} \{re^{i\theta} \mid \rho < r \leq 1 \wedge |\theta - \alpha| < \delta/2\}.$$

We now write the solution to the Dirichlet problem on the disk in a slightly different way that considered previously in this proof. To this end, let $P_r(\theta)$ be the *Poisson kernel*,

$$\text{Re} \left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right).$$

It is fairly well-known that if $\zeta \in \partial\mathbb{D}$ and $z \in \mathbb{D}$, then

$$\frac{1 - |z|^2}{|\zeta - z|^2} = \operatorname{Re} \left(\frac{\zeta + z}{\zeta - z} \right).$$

At the same time, if $z = re^{i\theta}$ and $\zeta = e^{i\theta'}$, then

$$\operatorname{Re} \left(\frac{\zeta + z}{\zeta - z} \right) = P_r(\theta - \theta').$$

Let f be a function on $\partial\mathbb{D}$ such that for each arc γ_j $f(\zeta) = f_j(\zeta)$ whenever ζ is a point in γ_j besides one of its endpoints. It then follows that when $0 < r < 1$,

$$u(re^{i\theta_1}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) P(\theta_1 - \theta) d\theta.$$

We can compute a rational number M such that

$$M > \max_k \max \operatorname{ran}(f_k).$$

Suppose α is such that for some j , $e^{i\alpha}$ is in γ_j but is not an endpoint. From the given data, we can enumerate all such α, j . We cycle through all such α, j as we scan the name of z . Fix a rational number $\epsilon > 0$. From ϵ and the given data, one can compute a rational $\delta_{\alpha, \epsilon} > 0$ such that the arc

$$\{e^{i\theta} : |\theta - \alpha| \leq \delta_{\alpha, \epsilon}\}$$

is contained in $\operatorname{ran}(\gamma_j)$ and such that

$$\frac{\epsilon}{3} > \max\{|f_j(e^{i\theta}) - f_j(e^{i\alpha})| : |\theta - \alpha| \leq \delta_{\alpha, \epsilon}\}.$$

We claim we can then compute $\rho_{\alpha, \epsilon}$ such that

$$\frac{\epsilon}{3M} > \max\{P_r(\theta) : |\theta| \geq \frac{1}{2}\delta \wedge \rho_{\alpha, \epsilon} \leq r \leq 1\}.$$

We postpone the computation of $\rho_{\alpha, \epsilon}$ so that we can reveal our intent. Namely, we claim that $|u(\zeta) - f(e^{i\alpha})| \leq \epsilon$ when $\zeta \in S(\rho_{\alpha, \epsilon}, \delta_{\alpha, \epsilon}, \alpha)$. For, let $\zeta \in S(\rho_{\alpha, \epsilon}, \delta_{\alpha, \epsilon}, \alpha)$, and write ζ as $re^{i\theta_1}$. Hence, $\rho < r \leq 1$, and we can choose θ_1 so that $|\theta_1 - \alpha| < \delta_{\alpha, \epsilon}/2$. If $r = 1$, then there is nothing more to do. So, suppose $r < 1$. For convenience, abbreviate $\delta_{\alpha, \epsilon}$ by δ . It then follows that

$$\begin{aligned} |u(\zeta) - f(e^{i\alpha})| &\leq \frac{1}{2\pi} \int_{|\theta - \alpha| \geq \delta} |f(e^{i\theta}) - f(e^{i\alpha})| P(\theta_1 - \theta) d\theta \\ &\quad + \frac{1}{2\pi} \int_{|\theta - \alpha| < \delta} |f(e^{i\theta}) - f(e^{i\alpha})| P(\theta_1 - \theta) d\theta. \end{aligned}$$

Suppose $|\theta - \alpha| \geq \delta$. Since $|\theta_1 - \alpha| < \delta/2$, $|\theta_1 - \theta| \geq \delta/2$ and so $P_r(\theta_1 - \theta) < \epsilon/3M$. Hence, the first term in the preceding sum is at most $2\epsilon/3$. At the same time, by our choice of $\delta_{\alpha, \epsilon}$, it follows that the second term is no larger than

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon}{3} P_r(\theta_1 - \theta) d\theta \leq \frac{\epsilon}{3}.$$

Hence, $|u(\zeta) - f(e^{i\alpha})| \leq \epsilon$.

Hence, as we scan the name of z , if we encounter a rational rectangle R and ϵ, α such that $S(\rho_{\alpha, \epsilon}, \delta_{\alpha, \epsilon})$ contains R , then we can list any subbasic neighborhood that contains $[f(e^{i\alpha}) - \epsilon, f(e^{i\alpha}) + \epsilon]$. It follows that if we only read neighborhoods that intersect the boundary, then we will write a name of $f(z)$ on the output tape.

We conclude by showing how to compute $\rho_{\alpha,\epsilon}$. Let $\delta' = \frac{1}{2}\delta$. The key inequality is

$$P_r(\theta) \leq P_r(\delta')$$

when $\delta' \leq |\theta| \leq \pi$. This is justified by Proposition 2.3.(c) on page 257 of [3]. Since $e^{i\delta'} \neq 1$, $P_1(\delta')$ is defined, and in fact is 0. We can compute $P_r(\delta')$ as a function of r on $[0, 1]$. We can thus compute $\rho_{\alpha,\epsilon}$ as required. \square

4. PROOF OF THE MAIN THEOREM

Theorem 4.1 (Effective Carathéodory Theorem). *From a name of a parameterization f of a Jordan curve J , and a name of a conformal map ϕ of the interior of J onto \mathbb{D} , it is possible to compute a name of the Carathéodory extension of ϕ .*

Proof. It follows from the main theorem of [7] that we can compute, uniformly from the given data, a name of the interior of J . It then follows that from the given data, we can compute a rational point z_0 in the interior of J . We can also compute a uniform local arcwise connectivity witness for J , h . We can assume h is increasing. Finally, from the given data, we can uniformly compute a sequence of rational polygonal Jordan curves $\{P_t\}_{t \in \mathbb{N}}$ such that $\|P_t - J\|_\infty < 2^{-t}$. See, for example, Lemma 6.1.10 of [13]. Hence, $J \subseteq D_{2^{-t}}(P_t)$. Let D denote the interior of J .

Let us allow ϕ to denote the Carathéodory extension of ϕ . The outline of our proof is as follows. We first compute the restriction of ϕ to the boundary of D . It then follows that we can compute ϕ^{-1} on $\partial\mathbb{D}$. It then follows by applying Lemma 3.2 to the real and imaginary parts of ϕ^{-1} that we can compute ϕ^{-1} on the closure of \mathbb{D} and hence ϕ on the closure of D .

To compute ϕ on the boundary of D , we appeal to the Principle of Type Conversion. Namely, we suppose are additionally given a name of a $\zeta_0 \in J$ and compute $\phi(\zeta_0)$. The key to this is to compute an arc Q from z_0 to ζ_0 such that $Q \subseteq \overline{D}$ and $Q \cap J = \{\zeta_0\}$. We do this by computing a sequence of rational polygonal arcs $\{Q_t\}_{t \in \mathbb{N}}$ as follows.

To compute Q_0 , we first compute a rational point $e_0 \neq z_0$ in the interior of J whose distance from ζ_0 is less than $2^{-h(0)}$. Now, D is an open connected set. It follows that there is a rational polygonal arc from z_0 to e_0 which is contained in the interior of J . Such an arc can now be discovered through a search procedure.

We now describe how we compute Q_{t+1} . This is to be a rational polygonal arc with the following properties.

- ((1)) Q_{t+1} is contained in the interior of J .
- ((2)) Q_{t+1} extends Q_t and has z_0 as a common endpoint.
- ((3)) The other endpoint of Q_{t+1} , e_{t+1} , is such that $|e_{t+1} - \zeta_0| < 2^{-h(t+1)-1}$.
- ((4)) The points in Q_{t+1} that are not in Q_t are contained in the disk of radius 2^{-t+1} about e_t .

By way of induction, suppose Q_t is a rational polygonal arc from z_0 to a point labelled e_t and that $|e_t - \zeta_0| < 2^{-h(t)-1}$. Suppose also that Q_t is contained in the interior of J . We first compute a rational point e_{t+1} in the interior of J and an integer r_{t+1} such that $|e_{t+1} - \zeta_0| < 2^{-h(t+1)-1}$, $e_{t+1} \in \mathbb{C} - \overline{D_{2^{-r_{t+1}}}(P_{r_{t+1}})}$, $|e_{t+1} - e_t| < 2^{-h(t)-1}$, and $e_{t+1} \notin Q_t$.

We summarize the key claim at this point in the proof by the following Lemma.

Lemma 4.2. e_{t+1} and e_t are in the same connected component of

$$D \cap D_{2^{-t}+2^{-h(t)-1}}(e_t).$$

Proof. For convenience, let $\epsilon = 2^{-t}$, $\delta = 2^{-h(t)}$, and $B = D_{\epsilon+\delta/2}(e_t)$.¹

By way of contradiction, suppose e_{t+1} and e_t are not in the same connected component of $D \cap B$. Let E and E' be the distinct components that contain e_t and e_{t+1} respectively. Since B is convex, there is a line segment l having e_t and e_{t+1} as endpoints and such that $l \subset B$. The length of l is of course less than $\delta/2$. Let $P = J \cap l \cap \partial E$, and let $P' = J \cap l \cap \partial E'$. Each of P and P' is closed and therefore compact as a subset of the plane. Therefore, there exists $p \in P$ and $p' \in P'$ such that $|p - p'| = d(P, P')$. Hence, $d(P, P') < \delta/2$.

For points, a and b of J , let $l(a, b)$ denote the minimum diameter of an arc in J having a and b as endpoints. Hence, $|a - b| \leq l(a, b)$.

Now, let A denote an arc in J such that the endpoints of A are p and p' and so that the diameter of A is $l(p, p')$. Note that $l(p, p') < \epsilon$. A straightforward connectivity argument shows that A intersects ∂B . (If not, then there is a piecewise linear arc with one endpoint in A and the other in E' - a contradiction.)

$A \cap \partial B$ is a compact subset of A . Hence, there is a point $q \in A \cap B$ such that the diameter of the subarc of A with endpoints p and q is minimal. Denote this diameter by $l_A(p, q)$. Then, we have

$$\begin{aligned} \epsilon + \delta/2 &= |e_t - q| \\ &\leq |e_t - p| \\ &\leq |e_t - p| + l_A(p, q) \\ &< |e_t - p| + l(p, p') \\ &< \delta/2 + \epsilon \end{aligned}$$

Which is a contradiction, and the Lemma is proved. \square

It then follows that there is a rational polygonal arc P from e_t to e_{t+1} that lies in the interior of J , does not cross Q_{t+1} , and that does not go further than $2^{-t} + 2^{-h(t)-1} \leq 2^{-t+1}$ from e_t . It now follows that a rational polygonal arc with properties (1) - (4) exists. Such a curve can be discovered through a simple search procedure which, in order to ensure (1), also computes s such that $Q_{t+1} \subseteq \mathbb{C} - \overline{D_{2^{-s}}(P_s)}$.

Let $Q = \bigcup_{t \in \mathbb{N}} Q_t$. It follows that one endpoint of Q is z_0 and the other is ζ_0 . By (1) and (2), $Q \cap J = \{\zeta_0\}$.

We have until now thought of each Q_t as a set. But, we wish to think of Q as a function. Furthermore, we wish to compute a name of a parameterization of Q from the given data. To this end, we now backtrack a little and describe in more detail how we parameterize each Q_t . Let $v_{t,1}, \dots, v_{t,n(t)}$ denote the vertices of Q_t in the order in which they are traversed so that $z_0 = v_{t,1}$ and $e_t = v_{t,n(t)}$. Set

$$a_{t,j} = \frac{j-1}{n(t)} \quad j = 1, \dots, n(t).$$

¹I wish to express here my gratitude to my colleague Dr. Dale Daniel for allowing me to use this proof, which is entirely of his own creation, in this paper.

Define Q_t to be the function on $[0, 1]$ that linearly maps each interval $[a_{t,j}, a_{t,j+1}]$ onto the line segment from $v_{t,j}$ to $v_{t,j+1}$, and that maps all of $[a_{t,n(t)}, 1]$ to the point $v_{t,n(t)}$.

It follows that $\lim_{t \rightarrow \infty} Q_t(x)$ exists for each $x \in [0, 1]$, and we define $Q(x)$ to be the value of this limit. Note that

$$\|Q_t - Q_{t+1}\|_\infty \leq 2^{-t} + 2^{-h(t)} \leq 2^{-t+1}.$$

It now follows that whenever $t' \geq t$,

$$\begin{aligned} \|Q_t - Q_{t'}\|_\infty &\leq \sum_{j=t}^{\infty} 2^{-j+1} \\ &= 2^{-t}. \end{aligned}$$

So, by Theorem 6.2.2.2 of [13], Q is computable uniformly from the given data.

Let A be the circle with center 0 and radius $1/2$. Let $A' = \phi^{-1}[A]$. Since $A \subseteq \mathbb{D}$, we can compute names of A' and its complement. We can then compute $R > 0$ such that the circle of radius R centered at ζ_0 does not intersect A' .

We now describe how we compute a name of $\phi(\zeta_0)$. Fix $r > 0$ such that $r < R, 1$. Compute a point w_1 on Q besides ζ_0 such that every point on Q between w_1 and ζ_0 inclusive lies in $D_r(\zeta_0)$. In fact, we can take w_1 to be a vertex of Q . We can then compute m such that

$$m^2 > \frac{(2\pi)^2}{\log(R) - \log(r)}.$$

Note that the right side of this inequality approaches 0 as r approaches 0 from the right. Let T_2 be the line $x = Re(\phi(w_1))$. Compute α such that $\phi(w_1) = |\phi(w_1)|e^{i\alpha}$. We will proceed under the assumption that T_2 hits A . For, we can apply the following construction and argument to $\psi = e^{-i\alpha}\phi$. Result will be an upper bound on $|\psi(w_1) - \psi(\zeta_0)| = |\phi(w_1) - \phi(\zeta_0)|$. Let T_1 be the line $x = Re(\phi(w_1)) + m$, and let T_3 be the line $x = Re(\phi(w_1)) - m$. We can thus assume m is small enough so that the lines T_1, T_3 both intersect A . Let p_1 be a point where one of these lines intersects $\partial\mathbb{D}$ and $\overline{p_1\phi(w_1)}$ does not hit A . Let $M = |p_1 - \phi(w_1)|$. We claim that $|\phi(w_1) - \phi(\zeta_0)| < 2M$. For, suppose otherwise. Then, when ϕ is applied to $\overline{w_1\zeta_0}$, the resulting arc hits T_2 and one of T_1, T_3 . Without loss of generality, suppose it hits T_1 . Let $S_k = \phi^{-1}[T_k]$ for $k = 1, 2$. Hence, S_1, S_2 hit $\overline{w_1\zeta_0}$. Hence, every circle centered at ζ_0 and whose radius is between r and R inclusive hits S_1 and S_2 . This puts us in position to use the “Length-Area Trick” as follows. Let $C_{r'}$ denote the circle with radius r' centered at ζ_0 . Fix $r \leq r' \leq R$. The curves S_1, S_2 have positive minimum distance from each other. It follows that there are points $z_{1,r'}$ and $z_{2,r'}$ on $C_{r'}$ that belong to S_1, S_2 respectively and such that no points on these curves appear on $C_{r'}$ between $z_{1,r'}$ and $z_{2,r'}$. (We do not claim that we can compute such points; this is not necessary to prove that our estimate is correct.) Let $K_{r'}$ denote the arc on $C_{r'}$ from $z_{1,r'}$ to $z_{2,r'}$. Hence, for some $\theta_{1,r'}, \theta_{2,r'}$

$$\begin{aligned} |\phi(z_{1,r'}) - \phi(z_{2,r'})| &= \left| \int_{K_{r'}} \phi'(z) dz \right| \\ &\leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)| r' d\theta \end{aligned}$$

On the other hand, $\phi(z_{1,r'})$ is on T_1 and $\phi(z_{2,r'})$ is on T_2 . Hence,

$$m \leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)| r' d\theta.$$

At the same time, by the Schwarz Integral Inequality (see *e.g.* Theorem 3.5, page 63 of [12]),

$$m^2 \leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta \int_{\theta_{1,r'}}^{\theta_{2,r'}} (r')^2 d\theta.$$

Whence

$$\begin{aligned} \frac{m^2}{r'} &\leq r' \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta \int_{\theta_{1,r'}}^{\theta_{2,r'}} d\theta \\ &\leq 2\pi r' \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta. \end{aligned}$$

If we now integrate both sides of this inequality with respect to r' from r to R , we obtain

$$m^2 [\log(R) - \log(r)] \leq 2\pi \int_r^R \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 r' d\theta dr'.$$

It follows from the Lusin Area Integral (see *e.g.* Lemma 13.1.2, page 386, of [8]) that this double integral is no larger than 2π . Hence,

$$m^2 \leq \frac{(2\pi)^2}{\log(R) - \log(r)}.$$

This is a contradiction. So, $|\phi(\zeta_0) - \phi(w_1)| < 2M$.

As r approaches 0 from the right, $\phi(w_1)$ approaches $\phi(\zeta_0)$ and so m, M can be chosen so as to approach 0. It follows that we can now generate a name of $\phi(\zeta_0)$.

We have now computed ϕ on ∂D . Since ϕ is injective, for each $z_0 \in \partial \mathbb{D}$, $\phi - z_0$ has a unique 0 on $\partial \mathbb{D}$. It follows from Corollary 6.3.5 of [13] that we can compute ϕ^{-1} on $\partial \mathbb{D}$. It now follows as noted in the introduction to this proof that we can compute ϕ^{-1} on $\overline{\mathbb{D}}$. For each $z_0 \in \overline{\mathbb{D}}$, $\phi^{-1} - z_0$ has a unique zero. Hence, we can compute ϕ on $\overline{\mathbb{D}}$. \square

In [9], P. Hertling showed that to compute a conformal map ϕ of a proper, simply connected domain D onto \mathbb{D} , it is necessary and sufficient to have a name of D and a name of the boundary of D as a closed set. This raises the question as to whether a name of the boundary of D as a closed set is, when D is Jordan, sufficient to compute the Carathéodory extension of D . In fact, this is not the case. For, from a Carathéodory extension of ϕ , it is possible to uniformly compute a parameterization of the boundary of D . But, it follows from Example 5.1 of [10] that there is a Jordan curve that is computable as a compact set but has no computable parameterization.

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